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THE NONLINEAR VERSION OF PAZY'S LOCAL EXISTENCE THEOREM

BY

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ABSTRACT

Let X be a real Banach space, $U \subset X$ a given open set, $A \subset X \times X$ a m-dissipative set and F: $C(0,a;U) \to L^\infty(0,a;X)$ a continuous mapping. Assume that A generates a nonlinear semigroup of contractions $S(t)$: $\overline{D(A)} \to \overline{D(A)}$, strongly continuous at the origin, with $S(t)$ compact for all $t > 0$. Then, for each $u_0 \in \overline{D(A)} \cap U$ there exists $T \in]0,a]$ such that the following initial value problem: $(du(t))/(dt) \in Au(t) + F(u)(t)$, $u(0) = u_0$, has at least one integral solution on $[0,T]$. Some extensions and applications are also included.

1. Introduction

The main result of this paper is a local existence theorem for integral solutions in the sense of Benilan and Brezis [7], to the initial value problem:

$$\frac{du(t)}{dt} \in Au(t) + F(u)(t), \qquad 0 \leqq t \leqq T, \tag{1.1}$$

$$u(0) = u_0, \tag{1.2}$$

where $A \subset X \times X$ is a m-dissipative set generating a nonlinear semigroup of contractions $S(t)$: $\overline{D(A)} \to \overline{D(A)}$, strongly continuous at the origin with $S(t)$ compact for all $t > 0$, while $U \subset X$ is an open set such that $u_0 \in \overline{D(A)} \cap U$ and F: $C(0,a;U) \to L^\infty(0,a;X)$ is a continuous mapping.

In the case in which A is linear and F is a point-wise defined function, i.e. F: $[0,a] \times U \to X$, the problem above was studied by Pazy [16] and the result obtained is concerned with mild solutions instead of integral solutions.

We note also the work of Attouch [1] who established an existence result for

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strong solutions of (1.1), (1.2) in the case in which X is a real Hilbert space, while A is the subdifferential of a proper, convex and lower semicontinuous function $\varphi: X \to]-\infty, +\infty]$ whose level sets are compact and in addition $F: [0, +\infty[\times X \to 2^X$ is a nonempty, convex and weakly compact valued mapping which is demi-closed and satisfies a certain growth condition.

Another paper, very closely related to the present one, is that of Crandall and Nohel [9] in which a general existence and uniqueness theorem is established for integral solutions to (1.1), (1.2) under an appropriate Lipschitz condition on F and more general assumptions on A.

Some recent extensions of Pazy's result in the linear case have been obtained firstly by Pavel [13], Schiaffino [17] and by Pavel and Vrabie [14].

We would like to point out that there is a very definite relation between our paper and the work of Barbu [2]. See also [4, ch. III, §3, theorem 3.1].

The reader is supposed to be familiar with the basic properties of m-dissipative sets in Banach spaces and with the main results in the theory of nonlinear semigroups of contractions. However, we recall for easy references the basic definitions and notations and we refer the reader to [4] for the relevant material on m-dissipative sets and nonlinear semigroups in general Banach spaces, material we shall use in the sequel.

In all that follows X is a real Banach space whose norm is denoted by $\|\cdot\|$ and X^* is its dual with the corresponding norm $\|\cdot\|_*$.

Let $G: X \to 2^{X^*}$ be the duality mapping, i.e.:

$$G(x) = \{x^* \in X^*; \langle x, x^* \rangle = \|x\|^2 = \|x^*\|_*^2\}, \tag{1.3}$$

for each $x \in X$. Let $(x, y) \in X \times X$ and define:

$$\langle y, x \rangle_S = \sup\{\langle y, x^* \rangle; x^* \in G(x)\}. \tag{1.4}$$

Noting that in the relations above $\langle u, u^* \rangle$ represents the value of $u^* \in X^*$ at $u \in K$, let us remark that $\langle \cdot, \cdot \rangle_S: X \times X \to R$ is an upper semicontinuous function. The proof of this simple fact may be found in [4, ch. I, §1, proposition 1.2].

If $A \subset X \times X$ and $x \in X$, denote by $Ax = \{y \in X; (x, y) \in A\}$. As usual, $D(A) = \{x \in X; Ax \neq \emptyset\}$ and $R(A) = \bigcup_{x \in D(A)} Ax$. Denote by $A^0x = \{y \in Ax; \|y\| \leqq \|z\| \text{ for all } z \in Ax\}$.

For each set $D \subset X$, $\bar{D}$ represents the closure of D while $\operatorname{conv} D$ denotes the convex hull of D. If $x \in X$ and $r > 0$, $B(x, r)$ is the closed ball with center x and radius r.

It is well known that a m-dissipative set $A \subset X \times X$ generates a nonlinear semigroup of contractions $S(t)$: $\overline{D(A)} \to \overline{D(A)}$, strongly continuous at the origin and for each $u \in \overline{D(A)}$ and $t \geqq 0$, the following exponential formula:

$$S(t)u = \lim_{n \to +\infty} \left(1 - \frac{t}{n} A\right)^{-n} u \tag{1.5}$$

holds. See for instance [8].

If $T > 0$, $C(0, T; X)$ denotes the space of all continuous functions from $[0, T]$ to X endowed with the uniform convergence topology, while $L^p(0, T; X)$ is the space of all equivalence classes of p-integrable functions from $[0, T]$ to X.

Following Benilan [5], [6], we define an integral solution of (1.1), (1.2) as a continuous function u: $[0, T] \to \overline{D(A)} \cap U$ such that $u(0) = u_0$ and satisfying the inequality:

$$\|u(\tilde{t}) - x\|^2 \leqq \|u(t) - x\|^2 + 2 \int_t^{\tilde{t}} \langle F(u)(s) + y, u(s) - x\rangle_s ds \tag{1.6}$$

for all $(x, y) \in A$ and $0 \leqq t \leqq \tilde{t} \leqq T$.

If $0 < T \leqq a$ and $u \in C(0, T; U)$ then $F(u)(t) = F(\tilde{u})(t)$ a.e. on $[0, T]$, where:

$$\tilde{u}(t) = \begin{cases} u(t) & \text{for } 0 \leqq t \leqq T \\ u(T) & \text{for } T \leqq t \leqq a. \end{cases}$$

It is also well known (see Benilan [5], [6]) that if $A \subset X \times X$ is a m-dissipative set, $h \in L^1(a, b; X)$, $u_0 \in \overline{D(A)}$, then the problem:

$$\frac{du(t)}{dt} \in Au(t) + h(t), \qquad a \leqq t \leqq b, \tag{1.7}$$

$$u(a) = u_0 \tag{1.8}$$

has a unique integral solution on $[a, b]$. Moreover, if u and v are two integral solutions of (1.7), (1.8) corresponding to $h \in L^1(a, b; X)$ and respectively to $g \in L^1(a, b; X)$, then:

$$\|u(\tilde{t}) - v(\tilde{t})\|^2 \leqq \|u(t) - v(t)\|^2 + 2 \int_t^{\tilde{t}} \langle h(s) - g(s), u(s) - v(s)\rangle_s ds \tag{1.9}$$

for all $a \leqq t \leqq \tilde{t} \leqq b$. See for instance [5], [6] and [4, ch. III, §2, theorem 2.1].

From (1.9) we deduce, using a standard argument, that:

$$\|u(\tilde{t}) - v(\tilde{t})\| \leqq \|u(t) - v(t)\| + \int_t^{\tilde{t}} \|h(s) - g(s)\| ds, \tag{1.10}$$

for all $a \leqq t \leqq \tilde{t} \leqq b$.

2. Statement of the main result

We begin with the hypothesis we need throughout the paper.

(H_1) *X is a real Banach space and $U \subset X$ is a given open set.*

(H_2) *$A \subset X \times X$ is a m-dissipative set which generates a nonlinear semigroup of contractions $S(t)$: $\overline{D(A)} \to \overline{D(A)}$, strongly continuous at the origin, with $S(t)$ compact for all $t > 0$.*

(H_3) *F: $C(0, a; U) \to L^\infty(0, a; X)$ is a continuous mapping.*

Now, we are able to formulate our main result:

THEOREM 2.1. *Assume that* (H_1), (H_2) *and* (H_3) *are satisfied. Then, for each $u_0 \in \overline{D(A)} \cap U$, there exists $T \in]0, a]$ such that* (1.1), (1.2) *has at least one integral solution on* $[0, T]$.

Let us observe that in the case in which A is linear, $h \in L^1(0, T; X)$ the class of integral solutions of (1.7), (1.8) coincides with the class of mild solutions, i.e. with the class of all continuous functions u: $[0, T] \to X$ verifying:

$$u(t) = S(t)u_0 + \int_0^t S(t-s)h(s)ds. \tag{2.1}$$

This follows from the fact that both integral solutions and mild solutions are uniform limits of strong solutions of certain approximate problems. Then, taking into account this remark and noting that in this last case $\overline{D(A)} = X$, we get from Theorem 2.1:

COROLLARY 2.1. *Assume that A is linear and* (H_1), (H_2) *and* (H_3) *are satisfied. Then, for each $u_0 \in U$, there exists $T \in]0, a]$ such that* (1.1), (1.2) *has at least one mild solution on* $[0, T]$.

Now, it is easy to see that Pazy's main result in [16] may be derived from our Corollary 2.1. We must only remark that each continuous function f: $[0, +\infty[\times U \to X$ generates a unique mapping F: $C(0, T; U) \to L^\infty(0, T; X)$ for all $T > 0$.

The proof of our main result, which will be developed in the next Section, is essentially based on Benilan's existence and uniqueness theorem mentioned in the Introduction, as well as on Crandall and Liggett's existence result in [8].

However, it should be added that the idea of the proof, which is in fact very simple, consists in the definition of an appropriate operator which satisfies the hypothesis of the well-known Schauder's Fixed Point Theorem, an operator whose fixed points are integral solutions of our problem (1.1), (1.2).

3. Proof of Theorem 2.1

Let $u_0 \in \overline{D(A)} \cap U$ and consider the positive numbers $r > 0$, $M > 0$ and $T > 0$ such that $B(u_0, r) \subset U$ and in addition:

$$\|F(u)(t)\| \leqq M, \qquad \text{a.e. on } [0, T], \tag{3.1}$$

for all $u \in C(0, T; U)$ with $u(t) \in B(u_0, r)$ for $0 \leqq t \leqq T$ and:

$$TM + \|S(t)u_0 - u_0\| \leqq r, \tag{3.2}$$

for all $0 \leqq t \leqq T$. Let us remark that we are able to choose such constants r, M and T to verify (3.1), (3.2) as F is locally bounded on $C(0, T; U)$ (being continuous) and $S(t)$ is strongly continuous at the origin.

Now, let K be the following subset of $C(0, T; U)$:

$$K = \{u \in C(0, T; U);\ u(t) \in B(u_0, r) \text{ for all } 0 \leqq t \leqq T\}. \tag{3.3}$$

Clearly, from (3.3) we deduce easily that K is a closed and convex subset of $C(0, T; X)$.

Let $v \in K$ and consider the following initial value problem:

$$\frac{du(t)}{dt} \in Au(t) + F(v)(t), \qquad 0 \leqq t \leqq T, \tag{3.4}$$

$$u(0) = u_0. \tag{3.5}$$

Then, by Benilan's existence and uniqueness theorem it follows that for each $v \in K$ the problem (3.4), (3.5) has a unique integral solution u. Define now the operator $Q: K \to C(0, T; U)$ by:

$$Qv = u, \tag{3.6}$$

where u, v satisfy together (3.4), (3.5), i.e. u is the integral solution of (3.4), (3.5).

It is obvious that (1.1), (1.2) has at least one integral solution iff the operator Q defined as above has at least one fixed point.

Thus, it appears very natural to study the properties of the operator Q. The next lemma is the crucial point in the proof of our Theorem 2.1.

Lemma 3.1. *The operator Q defined above maps K into K and is completely continuous.*

Proof of Lemma 3.1. From the definition of Q and from (1.10) it follows that:

$$\|Qv(t) - Qw(t)\| \leqq \int_0^t \|F(v)(s) - F(w)(s)\|\, ds. \tag{3.7}$$

But (3.7) and (H_3) show that Q is a continuous mapping from K into $C(0, T; U)$.

Now, we shall prove that for each $v \in K$, $Qv \in K$. To this end we shall apply (1.10) to Qv and to the integral solution w of the following problem:

$$\frac{dw(t)}{dt} \in Aw(t), \qquad 0 \leqq t \leqq T, \tag{3.8}$$

$$w(0) = u_0. \tag{3.9}$$

We note that such integral solution of (3.8), (3.9) may be expressed as:

$$w(t) = S(t)u_0. \tag{3.10}$$

Thus, according to (1.10), we get:

$$\| Qv(t) - S(t)u_0 \| \leqq \int_0^t \| F(v)(s) \| ds. \tag{3.11}$$

Combining (3.11) with:

$$\| Qv(t) - u_0 \| \leqq \| Qv(t) - S(t)u_0 \| + \| S(t)u_0 - u_0 \| \tag{3.12}$$

and taking into account that r, M and T verify (3.1), (3.2) we obtain:

$$\| Qv(t) - u_0 \| \leqq r. \tag{3.13}$$

But (3.13) implies that $QK \subset K$.

To complete the proof we have only to check out the equicontinuity of QK and the precompactness of the sets $\{Qv(t);\ v \in K\}$ for all $t \geqq 0$.

We begin with the precompactness of the sets defined above.

Let us observe that for $t = 0$ this is clearly the case, as the set considered is a singleton. Therefore, let $t > 0$ and $\varepsilon \in]0, t[$ and consider the following problem:

$$\frac{dv^\varepsilon(t)}{dt} \in Av^\varepsilon(t), \tag{3.14}$$

$$v^\varepsilon(t - \varepsilon) = Qv(t - \varepsilon). \tag{3.15}$$

Noting that v^ε and Qv are integral solutions for (3.14), (3.15) and respectively (3.4), (3.5) with the same initial data given at $t - \varepsilon$, we conclude from (1.10):

$$\| Qv(t) - v^\varepsilon(t) \| \leqq \int_{t-\varepsilon}^t \| F(v)(s) \| ds. \tag{3.16}$$

From (3.16) it follows that:

$$\| Qv(t) - v^\varepsilon(t) \| \leqq M\varepsilon. \tag{3.17}$$

Now, let us remark that $\{v^\varepsilon(t);\ v \in K\}$, where v^ε and v are related by (3.14), (3.15), is precompact in X as $v^\varepsilon(t) = S(\varepsilon)Qv(t-\varepsilon)$ and $S(\varepsilon)$ is compact by (H_2), while $\{Qv(t-\varepsilon);\ v \in K\}$ is bounded.

Therefore, from (3.17) and from the precompactness of $\{v^\varepsilon(t);\ v \in K\}$ we conclude that $\{Qv(t);\ v \in K\}$ is precompact in X for all $t \in [0, T]$.

For proving the equicontinuity of $\{Qv;\ v \in K\}$ on $[0, T]$, let us observe that (H_2) implies that for each bounded set $C \subset \overline{D(A)}$ the family of functions $\{S(\cdot)u;\ u \in C\}$ is equicontinuous at each $t > 0$.

Let $t > 0$ and $\varepsilon \in]0, t[$. From the remark above, taking into account that $v^\varepsilon(s) = S(s-t+\varepsilon)Qv(t-\varepsilon)$ for all $s \in [t-\varepsilon, t+\varepsilon]$, we may easily prove that $\{v^\varepsilon;\ v \in K\}$ is equicontinuous on $[t-\varepsilon, t+\varepsilon]$. Therefore, there exists $\delta(t,\varepsilon) > 0$ such that:

$$\|v^\varepsilon(s) - v^\varepsilon(t)\| \leqq M\varepsilon, \tag{3.18}$$

for all $s \in [0, T]$ with $|s - t| \leqq \delta(t, \varepsilon)$ and $v \in K$.

Furthermore, let us remark that:

$$\|Qv(s) - Qv(t)\| \leqq \|Qv(s) - v^\varepsilon(s)\| + \|v^\varepsilon(s) - v^\varepsilon(t)\| + \|v^\varepsilon(t) - Qv(t)\|. \tag{3.19}$$

From (3.17), (3.18) and (3.19) we get:

$$\|Qv(s) - Qv(t)\| \leqq 3M\varepsilon, \tag{3.20}$$

for all $s \in [0, T]$ with $|s - t| \leqq \delta(t, \varepsilon)$ and $v \in K$.

As from (3.11), (3.12) we may derive the equicontinuity of $\{Qv; v \in K\}$ at $t = 0$; the last relation (3.20) and the remark above imply the equicontinuity of the family $\{Qv; v \in K\}$ on the whole interval $[0, T]$.

Therefore, QK satisfies the hypothesis of the well-known Arzela and Ascoli's Theorem and thus QK is relatively compact in $C(0, T; X)$ and this completes the proof of Lemma 3.1.

Now, for proving Theorem 2.1, we have only to remark that Q has at least one fixed point (by Lemma 3.1 and Schauder's Fixed Point Theorem) and in addition that this fixed point u is an integral solution to (1.1), (1.2) on $[0, T]$, as claimed.

In the next Section we shall treat the case in which F is a given multivalued mapping.

4. The multivalued case

In all what follows F: $[0, +\infty[\times U \to 2^X$ is a multivalued mapping, while X and U are as above.

Firstly, we need the following definitions:

A mapping F: $[0,+\infty[\times U \to 2^X$ is called *sequentially demi-closed* if for each sequence $\{(t_n, x_n, y_n)\}_{n=1}^{\infty} \subset [0,+\infty[\times U \times X$ with $y_n \in F(t_n, x_n)$ for $n = 1,2,\cdots$, (t_n, x_n) strongly convergent to (t_0, x_0) belonging to $[0,+\infty[\times U$ and y_n weakly convergent to y_0, it follows that $y_0 \in F(t_0, x_0)$.

A mapping F: $[0,+\infty[\times U \to 2^X$ is called *locally bounded* if for each $(t,x) \in [0,+\infty[\times U$ there exist $\delta > 0$ and $M > 0$, such that:

$$\|F(s,y)\| \leqq M, \tag{4.1}$$

for all $(s,y) \in [0,+\infty[\times U$ with $\|x - y\| \leqq \delta$ and $|t-s| \leqq \delta$.

Here, $\|F(s,y)\|$ is defined by:

$$\|F(s,y)\| = \sup\{\|z\|;\ z \in F(s,y)\}. \tag{4.2}$$

A continuous function u: $[0,T] \to \overline{D(A)} \cap U$ is called integral solution for (1.1), (1.2) if there exists a function $g \in L^1(0,T;X)$, such that $g(s) \in F(s,u(s))$ a.e. on $[0,T]$ and u is an integral solution for the following problem:

$$\frac{du(t)}{dt} \in Au(t) + g(t), \qquad 0 \leqq t \leqq T, \tag{4.3}$$

$$u(0) = u_0. \tag{4.4}$$

Now, we are able to formulate the hypothesis we shall use in this Section, as well as the main result in the multivalued case.

(I_1) *X^* is uniformly convex and $U \subset X$ is a given open set.*

(I_2) *$A \subset X \times X$ satisfies* (H_2).

(I_3) *F: $[0,+\infty[\times U \to 2^X$ is a nonempty, convex and bounded valued mapping which is sequentially demi-closed and locally bounded.*

THEOREM 4.1. *Assume that* (I_1), (I_2) *and* (I_3) *are satisfied. Then, for each $u_0 \in \overline{D(A)} \cap U$, there exists $T \in [0,+\infty[$ such that* (1.1), (1.2) *has at least one integral solution on* $[0,T]$.

Before proceeding to the proof of Theorem 4.1, we shall prove a technical result which is the main argument we shall use. A slightly different form of this result may be found in [11, lemma 8.2].

LEMMA 4.1. *Let X be a real, reflexive Banach space and let $\{u_n\}_{n=1}^{\infty}$ be a sequence belonging to $L^p(0,T;X)$, $p > 1$, such that $\{u_n(t)\}_{n=1}^{\infty}$ is bounded for almost all $t \in [0,T]$. Let $Y(t)$ be the set of all weak cluster points of $\{u_n(t)\}_{n=1}^{\infty}$. If u_n converges weakly in $L^p(0,T;X)$ to u, then:*

(4.5) $$u(t) \in \overline{\operatorname{conv} Y(t)} \text{ almost everywhere on } [0, T].$$

We shall indicate the weak convergence by "$\rightharpoonup$" and the strong convergence by "$\rightarrow$".

PROOF OF LEMMA 4.1. Since $u_n \rightharpoonup u$ in $L^p(0, T; X)$ there exists a sequence $\{v_n\}_{n=1}^{\infty}$ of convex combinations of $\{u_n\}_{n=1}^{\infty}$ strongly convergent to u, such that the indices of u_k that appears in v_n satisfy $k \geqq n$.

Then, let us observe that there exists a subsequence $\{v_{n_k}\}_{k=1}^{\infty}$ of $\{v_n\}_{n=1}^{\infty}$ convergent almost everywhere to u. We fix $t \in [0, T]$ and consider an arbitrary open half space E of X containing $Y(t)$. Obviously, $u_n(t) \in E$ for n large enough and this implies that $v_{n_k}(t) \in E$ for k large enough. Thus, $u(t) \in \bar{E}$ for almost all $t \in [0, T]$. As $\overline{\operatorname{conv} Y(t)}$ is the intersection of all closed half spaces which contain it, we conclude that $u(t) \in \overline{\operatorname{conv} Y(t)}$ a.e. on $[0, T]$ and this completes the proof of Lemma 4.1.

5. Proof of Theorem 4.1

The idea of the proof consists in the construction of a sequence of approximate solutions of (1.1), (1.2) satisfying the hypothesis of the infinite dimensional version of Arzela and Ascoli's Theorem, the techniques used being those of Barbu [2]. See also [4, ch. III, §3, lemma 3.1].

Let $u_0 \in \overline{D(A)} \cap U$ and consider the positive numbers $r > 0$, $M > 0$ and $T > 0$ such that $B(u_0, r) \subset U$ and in addition:

(5.1) $$\|F(t, u)\| \leqq M,$$

for all $(t, u) \in [0, T] \times B(u_0, r)$,

(5.2) $$TM + \|S(t)u_0 - u_0\| \leqq r,$$

for all $t \in [0, T]$.

The meaning of $\|F(t, u)\|$ in (5.1) is that of (4.2). Let us observe that we are able to choose the constants r, M and T verifying (5.1) and (5.2) as F is locally bounded on U by (I_3) and $S(t)$ is strongly continuous at the origin.

For every positive integer n, consider the partition Δ_n of $[0, T]$ given by: $0 = t_0^n < t_1^n < t_2^n < \cdots < t_{N(n)}^n = T$ with $t_{i+1}^n - t_i^n \leqq 1/n$.

Let $u_n \colon [0, T] \to X$ be the continuous function inductively defined by:

(5.3) $$u_n(t) = v_n^i(t - t_{i-1}^n) \qquad \text{for } t_{i-1}^n \leqq t \leqq t_i^n,$$

where v_n^i is the integral solution of the following problem:

$$\frac{dv_n^i(t)}{dt} \in Av_n^i(t) + g(t_{i-1}^n, u_n(t_{i-1}^n)), \tag{5.4}$$

$$v_n^i(0) = u_n(t_{i-1}^n). \tag{5.5}$$

In (5.4) $g(t_{i-1}^n, u_n(t_{i-1}^n))$ is a fixed element of $F(t_{i-1}^n, u_n(t_{i-1}^n))$.

Take $u_n(0) = u_0$ for all $n = 1, 2, \cdots$ and define the step function: $a_n(t) = t_{i-1}^n$ for $t \in [t_{i-1}^n, t_i^n[$, $i = 1, 2, \cdots, N(n)$.

Let us observe that $|a_n(t) - t| \leqq 1/n$ for all $t \in [0, T]$ and thus $\lim_{n\to\infty} a_n(t) = t$ uniformly on $[0, T]$.

Clearly, as v_n^i exists and is unique by Benilan's existence and uniqueness theorem, (5.4), (5.5) imply that u_n is the integral solution of the following problem:

$$\frac{du_n(t)}{dt} \in Au_n(t) + g_n(t), \qquad 0 \leqq t \leqq T, \tag{5.6}$$

$$u_n(0) = u_0, \tag{5.7}$$

where $g_n(t) = g(t_{i-1}^n, u_n(t_{i-1}^n))$ on $[t_{i-1}^n, t_i^n[$, $i = 1, 2, \cdots, N(n)$.

Let us remark that:

$$g_n(t) \in F(a_n(t), u_n(a_n(t))), \tag{5.8}$$

for all $t \in [0, T]$.

Now, using the same arguments as in the proof of Lemma 3.1, we may prove that the sequence $\{u_n\}_{n=1}^\infty$ is relatively compact in $C(0, T; X)$ and in addition that $u_n(t) \in B(u_0, r) \cap \overline{D(A)}$ for all $t \in [0, T]$.

We leave this simple fact to the reader and we shall prove that we may choose a subsequence $\{u_{n_k}\}_{k=1}^\infty$ of $\{u_n\}_{n=1}^\infty$ strongly convergent in $C(0, T; X)$ to an integral solution of (1.1), (1.2).

To this end, let us observe that $\{g_n\}_{n=1}^\infty$ is uniformly bounded and therefore, via Alaoglu's Theorem (see for instance [10, ch. V, §4.2, theorem 2]), we conclude that $\{g_n\}_{n=1}^\infty$ is weakly compact in $L^2(0, T; X)$.

Consider now a subsequence $\{g_{n_k}\}_{k=1}^\infty$ of $\{g_n\}_{n=1}^\infty$ weakly convergent in $L^2(0, T; X)$ to some function g and let us choose also a strongly convergent subsequence $\{u_{n_{k_j}}\}_{j=1}^\infty$ of $\{u_{n_k}\}_{k=1}^\infty$. Denote by u the limit of the subsequence above in $C(0, T; X)$. We shall prove that:

$$g(t) \in F(t, u(t)), \tag{5.9}$$

for almost all $t \in [0, T]$.

One has only to observe that the set $W(t)$ of all weak cluster points of

$\{g_{n_k}\}(t)\}_{k=1}^{\infty}$ is nonempty (as $\{g_{n_k}(t)\}_{k=1}^{\infty}$ is bounded and X is reflexive) and $W(t) \subset F(t, u(t))$. This simple fact follows from the sequentially demi-closedness assumption (I_3).

Then, we are in the hypothesis of Lemma 4.1 and we may conclude that (5.9) holds.

Finally it is only an exercise to prove that u is an integral solution of (1.1), (1.2) and this completes the proof of Theorem 4.1.

6. Continuation of the solution: applications

In this Section we shall formulate some results about the continuation of the solutions in the two cases considered and, as the proof follows, by using standard arguments, we shall not give details.

Moreover, we shall prove a very simple sufficient condition for (H_2) in the nonlinear case. For the linear case the reader is referred to Pazy [15] where a characterization of the compact semigroups is given.

We start with the following:

THEOREM 6.1. *Assume that* (H_1), (H_2) *and* (H_3) *are satisfied and in addition that F maps bounded sets in $C(0, T; U)$ into bounded sets in $L^{\infty}(0, T; X)$ for all fixed $T \in]0, a]$. Then, for each $u_0 \in \overline{D(A)} \cap U$ there exists an integral solution u of* (1.1), (1.2) *defined on a maximal interval of existence $[0, T_{\max}[$, where either $T_{\max} = a$, or if $T_{\max} < a$, then the solution u is unbounded on $[0, T_{\max}[$.*

THEOREM 6.2. *Assume that* (I_1), (I_2) *and* (I_3) *are satisfied and in addition that F maps bounded sets in $[0, a] \times U$ into bounded sets in X. Then, for each $u_0 \in \overline{D(A)} \cap U$ there exists an integral solution u of* (1.1), (1.2) *defined on a maximal interval of existence $[0, T_{\max}[$, where either $T_{\max} = a$, or if $T_{\max} < a$, then the solution u is unbounded on $[0, T_{\max}[$.*

The following corollaries are also very useful in applications.

COROLLARY 6.1. *Assume that* (H_1), (H_2) *are satisfied and in addition that F: $[0, +\infty[\times U \to X$ is a continuous mapping.*

Assume further that:

$$\|F(t, x)\| \leqq K_1(t)\|x\| + K_2, \tag{6.1}$$

for each $(t, x) \in [0, +\infty[\times U$, where $K_1 \in L^{\infty}_{\text{loc}}(0, +\infty; R)$ and $K_2 \in R$.

Then, for each $u_0 \in \overline{D(A)} \cap U$ there exists an integral solution u of (1.1), (1.2) *defined on the whole positive half axis.*

COROLLARY 6.2. *Assume that* (I_1), (I_2) *and* (I_3) *are satisfied and in addition that*:

$$\|F(t,x)\| \leqq K_1(t)\|x\| + K_2 \tag{6.2}$$

for each $(t,x) \in [0, +\infty[\times U$ *where* $K_1 \in L^\infty_{loc}(0, +\infty; R)$ *and* $K_2 \in R$.

Then, for each $u_0 \in \overline{D(A)} \cap U$ *there exists an integral solution* u *of* (1.1), (1.2) *defined on the whole positive half axis.*

Now, we shall indicate a large class of problems for which we cannot use any of the existence results mentioned in the Introduction, but for which we are able to apply our Theorem 2.1.

We begin with the following sufficient condition for (H_2).

LEMMA 6.1. *Let* $A \subset X \times X$ *be a* m*-dissipative set and assume that the nonlinear semigroup of contractions* $S(t): \overline{D(A)} \to \overline{D(A)}$, *generated by* A *satisfies*:

(i) *For each* $t > 0$, $S(t)\overline{D(A)} \subset D(A)$.

(ii) *For each* $t > 0$, $A^0S(t)$ *maps bounded sets in* $\overline{D(A)}$ *into bounded sets in* X.

(iii) *The resolvent* $(1 - \lambda A)^{-1}$ *is compact for some* $\lambda > 0$.

Then, $S(t)$ *is compact for all* $t > 0$.

PROOF OF LEMMA 6.1. One has only to observe that for each $x \in \overline{D(A)}$, $S(t)x = (1 - \lambda A)^{-1}(1 - \lambda A^0)S(t)x$ and this completes the proof of Lemma 6.1.

We would like to point out that in the case in which $-A$ is the subdifferential of a given proper, convex and lower semicontinuous function $\varphi: X \to]-\infty, +\infty]$, X being a real Hilbert space, conditions (i) and (ii) are automatically satisfied [4, ch. IV, §2, theorem 2.1 and theorem 2.3] and in this case Lemma 6.1 is due to Konishi [12]. It is easy to see that in this case (iii) holds iff the level sets $\{x \in X;\ \|x\|^2 + \varphi(x) \leqq M\}$ are relatively compact for all $M > 0$.

We shall prove in what follows that (i) and (ii) hold not only for subdifferential operators but also for other types of m-dissipative operators.

To this end let H be a real Hilbert space and V a real reflexive Banach space such that $V \subset H$, the inclusion mapping of V into H being *completely continuous.* Suppose that V is dense in H and denote by V' the dual of V. Then, it is obvious that $V \subset H \subset V'$.

Denote by $\|\cdot\|$, $|\cdot|$ and by $\|\cdot\|_*$ the norms in V, H and respectively in V' and by $\langle \cdot, \cdot \rangle$ the pairing between V and V'.

Let $A: V \to V'$ be a single valued operator which satisfies:

(J_1) A *is semicontinuous* i.e. *the mapping* $t \mapsto \langle A(u + tv), w \rangle$ *is continuous on* $[0, +\infty[$ *for all* $u, v, w \in V$.

(J_2) *$\langle Au - Av, u - v\rangle \geqq \omega \| u - v \|^2$ for all $u, v \in V$, where $\omega > 0$ is a fixed constant.*

(J_3) *$\| Au \|_* \leqq C_1(\| u \| + 1)$ for all $u \in V$, where C_1 does not depend on u.*

Now, we are able to formulate the main result of this Section.

THEOREM 6.3. *Assume that $V \subset H \subset V'$ and the inclusion $V \subset H$ is completely continuous and V is dense in H. Assume further that* (J_1), (J_2) *and* (J_3) *are satisfied. Then, the set $A_H \subset H \times H$ defined by:*

$$A_H u = - Au \tag{6.3}$$

for $u \in D(A_H) = \{u \in V;\ Au \in H\}$, is m-dissipative and generates a nonlinear semigroup of contractions $S(t)$: $H \to H$ strongly continuous at the origin with $S(t)$ compact for all $t > 0$.

PROOF OF THEOREM 6.3. The fact that A_H *is* m-dissipative is well known. See for instance [4, ch. III, §2, theorem 2.5].

Therefore, we shall prove that the semigroup generated by A_H satisfies the hypothesis of Lemma 6.1. Let us observe that (i) follows from [3, theorem 1]. Now, it is only an exercise to prove that $(1 - A_H)^{-1}$ is compact, as A is coercive (by (J_2)) and the inclusion $V \subset H$ is completely continuous.

To conclude we have to remark that (ii) is implied by:

$$| tAS(t)u | \leqq C | u |, \tag{6.4}$$

for all $t > 0$, $u \in H$, where $C > 0$ does not depend on $(t, u) \in [0, +\infty[\times H$, a relation which is an immediate consequence of formula (3.9) in [3] and this completes the proof of Theorem 6.3.

We conclude the paper with the following example which is important by itself.

EXAMPLE 6.1. Let $\Omega \subset R^n$ be an open and bounded set whose boundary Γ is a C^∞ manifold, and consider the following differential operator

$$Au = \sum_{|k| \leqq m} (-1)^k D^k A_k (x, u, Du, \cdots, D^m u), \tag{6.5}$$

where $A_k : \Omega \times R^m \to R$ satisfy:

(K_1) *A_k is measurable in x and continuous in the rest of the variables. There exist $p > 0$, $g \in L^q(\Omega)$, $1/p + 1/q = 1$ and $C > 0$ such that:*

$$| A_k (x, u) | \leqq C(| u |^{p-1} + g(x)) \qquad \text{a.e. } x \in \Omega. \tag{6.6}$$

(K_2) *There exists* $\omega > 0$ *such that for all* $(u, v) \in R^{\bar{m}} \times R^{\bar{m}}$ *and for almost all* $x \in \Omega$ *the following inequality*:

$$\sum_{|k| \leq m} (A_k(x, u) - A_k(x, v))(u_k - v_k) \geqq \omega \sum_{|k| \leqq m} |u_k - v_k|^2 \tag{6.7}$$

holds.

Consider the nonlinear integro-differential equation:

$$\frac{\partial u}{\partial t} + \sum_{|k| \leqq m} (-1)^k D^k A_k(x, u, \cdots, D^m u) + \int_0^t a(t-s) f(s, u(s)) ds = 0, \tag{6.8}$$

with boundary conditions:

$$D^k u = 0 \quad \text{on} \quad [0, T] \times \Gamma \qquad \text{for } |k| \leqq m - 1 \tag{6.9}$$

and initial condition:

$$u(0, x) = u_0(x) \qquad \text{on } \Omega. \tag{6.10}$$

THEOREM 6.4. *Assume that* (K_1) *and* (K_2) *are satisfied and in addition that* $a: [0, +\infty[\to R$ *and* $f: [0, +\infty[\times R \to R$ *are continuous functions. Assume further that*:

$$|f(t, u)| \leqq G_1(t) |u| + G_2 \tag{6.11}$$

where $G_1 \in L^\infty_{\mathrm{loc}}(0, +\infty; R)$ *and* $G_2 \in R$.

Then, for each $u_0 \in L^2(\Omega)$ *there exists* $T > 0$ *such that* (6.8), (6.9) *has at least one integral solution on* $[0, T]$.

PROOF OF THEOREM 6.4. Take $V = W_0^{m,p}(\Omega)$, $H = L^2(\Omega)$ and let us observe that $A: W_0^{m,p}(\Omega) \to W^{-m,q}(\Omega)$ defined by (6.5) satisfies the hypothesis of Theorem 6.3. Now, let us remark that (6.8), (6.9) (6.10) may be rewritten in the following form:

$$\frac{du(t)}{dt} \in Au(t) + F(u)(t), \qquad 0 \leqq t \leqq T, \tag{6.12}$$

$$u(0) = u_0, \tag{6.13}$$

where:

$$F(u)(t) = -\int_0^t a(t-s) f(s, u(s)) ds. \tag{6.14}$$

It is obvious that for each $T > 0$, F is a continuous mapping from

$C(0, T; L^2(\Omega))$ into $C(0, T; L^2(\Omega))$ and therefore we are in the hypothesis of Theorem 2.1, thereby completing the proof of Theorem 6.4.

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